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Kumamoto University
Meromorphic approximation theorem with respect to a semigroup of holomorphic line bundles in a Stein space

Makoto Abe*

Abstract. We give a meromorphic approximation theorem with respect to a semigroup of holomorphic line bundles in a reduced Stein space, which generalizes both Théorème 2 of Hirschowitz (Ann. Scuola Norm. Sup. Pisa (3) 25:47–58, 1971) and Theorem 11 of Abe (Ann. Mat. Pura Appl. (4) 184:263–274, 2005).

1. Introduction

Let $K$ be an arbitrary rationally convex compact set of $\mathbb{C}^n$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\epsilon > 0$ there exists a rational function $h$ on $\mathbb{C}^n$ such that $h$ is holomorphic near $K$ and $\|\varphi - h\|_K < \epsilon$. Here we denote by $\mathcal{O}(K)$ the set of functions holomorphic near $K$. This fact is known as the rational approximation theorem of Weil-Oka.

In the literature there are two different definitions of meromorphically convex hull of a compact set $K$ of a reduced complex space $X$. According to the notation of Hirschowitz [11] the one is $hK_X$ and the other is $H K_X$ (see Sect. 2). In general, there can exist a compact set $K$ of $X$ such that $hK_X \subset

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If $X$ is a Stein manifold, then Colțoiu [5] proved that $hK_X = \mathcal{H}K_X$ for every compact set $K$ of $X$ if and only if $\text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) = 0$. If $X = \mathbb{C}^n$, then the set $hK_{\mathbb{C}^n} = \mathcal{H}K_{\mathbb{C}^n}$ moreover coincide with the rationally convex hull of $K$ in $\mathbb{C}^n$.

Hirschowitz [11, Théorème 2] obtained the following meromorphic approximation theorem which generalizes the rational approximation theorem of Weil-Oka. Let $X$ be a Stein manifold and $K$ a compact set of $X$ such that $hK_X = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exists a meromorphic function $h \in \mathcal{M}(X) \cap \mathcal{O}(K)$ such that $\|\varphi - h\|_K < \varepsilon$.

On the other hand, Abe [1, Theorem 11] proved the following meromorphic approximation theorem, which also generalizes the rational approximation theorem of Weil-Oka. Let $X$ be a reduced Stein space and $K$ a compact set of $X$ such that $hK_X = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist holomorphic functions $f, g \in \mathcal{O}(X)$ such that the set $\{g = 0\}$ is nowhere dense in $X$, $g \neq 0$ on $K$, and $\|\varphi - (f/g)\|_K < \varepsilon$.

In this paper we prove a meromorphic approximation theorem with respect to a semigroup of holomorphic line bundles in a reduced Stein space, which includes both Hirschowitz [11, Théorème 2] and Abe [1, Theorem 11] as two special cases. The main results in this paper were announced without proof in Abe [3].

We first define the generalized meromorphically convex hull $\tilde{K}_{X,G}$ of a compact set $K$ of a reduced complex space $X$ with respect to a subsemigroup $G$ of the Picard group $\text{Pic}(X)$ of $X$, that is, a set $G$ of holomorphic line bundles on $X$ which is closed under the tensor product (see Sect. 2). Then the main result is as follows.

Let $X$ be a reduced Stein space and $G$ a subsemigroup of $\text{Pic}(X)$. Let $K$ be a compact set of $X$ such that $\tilde{K}_{X,G} = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist $L \in G$ and holomorphic sections $f, g \in \Gamma(X, \mathcal{O}(L))$ such that the set $\{g = 0\}$ is nowhere dense in $X$, $g \neq 0$ on $K$, and $\|\varphi - (f/g)\|_K < \varepsilon$ (see Theorem 5.1).

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2. Preliminaries

Throughout this paper complex spaces are always assumed to be reduced and second countable. We denote by $\mathcal{O}$ (resp. $\mathcal{M}$) the additive sheaf of germs of holomorphic (resp. meromorphic) functions and by $\mathcal{O}^*$ the multiplicative sheaf of invertible germs of holomorphic functions on a complex space.

Let $X$ be a complex space and $K$ a subset of $X$. Then the set

$$K_x = \{ x \in X \mid f(x) \in f(K) \text{ for every } f \in \mathcal{O}(X) \}$$

is said to be the \textit{meromorphically convex hull} of $K$ in $X$. We use hereafter the notation $K_x$ of Lupacciolu [14] instead of the notation $\mu K_X$ of Hirschowitz [11, p. 49]. The set $K_x$ is contained in the \textit{holomorphically convex hull}

$$K_x = \{ x \in X \mid |f(x)| \leq \|f\|_K \text{ for every } f \in \mathcal{O}(X) \}$$

of $K$ in $X$. The set $K_x$ is closed in $X$ if $K$ is compact. Therefore the set $K_x$ is compact if $X$ is holomorphically convex and $K$ is compact.

Let $X$ be a complex space. The group $\text{Pic}(X) := H^1(X, \mathcal{O}^*)$ is said to be the \textit{Picard group} of $X$ and is identified with the set of holomorphic line bundles on $X$. Let $G$ be a subsemigroup of $\text{Pic}(X)$, that is, a set of holomorphic line bundles on $X$ which is closed under the tensor product. Let $K$ be a subset of $X$. Then the set

$$K_{x,G} = \{ x \in X \mid \text{for every } L \in G \text{ and for every } s \in \Gamma(X, \mathcal{O}(L)) \text{ such that } s(x) = 0 \text{ we have that } \{s = 0\} \cap K \neq \emptyset \}$$

is said to be the (generalized) \textit{meromorphically convex hull} of $K$ in $X$ with respect to $G$. We have that $\tilde{K}_{x,G} = \bigcap_{\mathcal{A} \in \mathcal{A}(G,K)} (X \setminus A)$, where

$$\mathcal{A}(G,K) := \{ A \mid \text{there exist } L \in G \text{ and } s \in \Gamma(X, \mathcal{O}(L)) \text{ such that } A = \{s = 0\} \text{ and } A \cap K = \emptyset \}.$$ 

For simplicity we write

$$\tilde{K}_{x,L} := \tilde{K}_{x,\{L^\nu \mid \nu \in \mathbb{N}\}} \text{ and } \mathcal{A}(L,K) := \mathcal{A}(\{L^\nu \mid \nu \in \mathbb{N}\}, K).$$
for every $L \in \text{Pic}(X)$. Let $1_X$ denote the unit element of $\text{Pic}(X)$, that is, the trivial holomorphic line bundle on $X$. We have that

$$\tilde{K}_X = \tilde{K}_X,1_X = \tilde{K}_X,\{1_X\}.$$ 

We have the following proposition, the proof of which is easy and is omitted.

**Proposition 2.1.** Let $X$ be a complex space and $G$ a subsemigroup of $\text{Pic}(X)$. Let $K$ be a subset of $X$ and let $M := \tilde{K}_{X,G}$. Then we have that $\tilde{M}_{X,G} = M$.

A subset $A$ of a complex space $X$ is said to be a **hypersurface** in $X$ if there exists a coherent principal analytic ideal sheaf $\mathcal{I}$ on $X$ such that $N(\mathcal{I}) = A$ and $A$ is nowhere dense in $X$. If $X$ is a complex manifold, then a subset $A$ of $X$ is a hypersurface in $X$ if and only if $A$ is an analytic set of $X$ such that $\text{codim}_x A = 1$ for every $x \in A$ (see Hitotumatu [12, p. 155] or Taylor [25, p. 97]).

Let $X$ be a complex space and $K$ a subset of $X$. According to the notation of Hirschowitz [11, p. 50] let

$$h_K := \{x \in X \mid \text{for every hypersurface } A \text{ in } X \text{ such that } x \in A \text{ we have that } A \cap K \neq \emptyset \}.$$

Then we have that $h_K = \bigcap_{A \in \mathcal{H}(K)} (X \setminus A)$, where $\mathcal{H}(K)$ denotes the set of hypersurfaces in $X$ which does not intersect $K$.

Since every coherent principal analytic ideal sheaf $\mathcal{I}$ on a complex space $X$ such that $N(\mathcal{I})$ is nowhere dense in $X$ defines a holomorphic line bundle $L$ on $X$ and a global section $s \in \Gamma(X, \mathcal{O}(L))$ such that $\{s = 0\} = N(\mathcal{I})$, we have that $\mathcal{H}(K) \subset \mathcal{J}(\text{Pic}(X), K)$ for every $K \subset X$.

**Proposition 2.2.** Let $X$ be a Stein space without isolated points. Then for every subset $K$ of $X$ we have that $h_K = \tilde{K}_{X,\text{Pic}(X)}$.

**Proof.** Since $\mathcal{H}(K) \subset \mathcal{J}(\text{Pic}(X), K)$, we have that $\tilde{K}_{X,\text{Pic}(X)} \subset h_K$. Let $x \notin \tilde{K}_{X,\text{Pic}(X)}$. Then there exist $L \in \text{Pic}(X)$ and $s \in \Gamma(X, \mathcal{O}(L))$ such that $x \in A := \{s = 0\}$ and $A \cap K = \emptyset$. Let $\Lambda := \{\lambda \in \Lambda \mid s \neq 0 \text{ on } X_\lambda\}$, $\Lambda'' := \Lambda \setminus \Lambda'$, and $X' := \bigcup_{\lambda \in \Lambda'} X_\lambda$. Take a point $p_\lambda \in \left(X_\lambda \setminus \left(\bigcup_{\mu \neq \lambda} X_\mu\right)\right)$
{x} for every $\lambda \in \Lambda''$. Then $Z := X' \cup \{x\}$, $P := \{p_\lambda \mid \lambda \in \Lambda''\}$, and $S := Z \cup P$ are analytic sets of $X$. By the Cartan theorem B the restriction map

$$\Gamma(X, \mathcal{O}(L)) \to \Gamma(X, \mathcal{O}(L)/i(S) \cdot \mathcal{O}(L)) \cong \Gamma(Z, \mathcal{O}(L|_Z)) \oplus \mathbb{C}^P$$

is surjective, where $i(S)$ denotes the maximal defining ideal sheaf of $S$. Therefore there exists $t \in \Gamma(X, \mathcal{O}(L))$ such that $t|_Z = s|_Z$ and $t(p_\lambda) \neq 0$ for every $\lambda \in \Lambda''$. Let $\mathcal{I}$ be the analytic ideal sheaf defined by $t$. Then $\mathcal{I}$ is principal at every point of $X$ and the set $B := \{t = 0\} = N(\mathcal{I})$ is nowhere dense in $X$. Since $K \cup \{x\} \subset Z$, we have that $x \in B$ and $B \cap K = \emptyset$. It follows that $x \not\in hK_X$. Thus we proved that $hK_X \subset K_{X,G} \subset K_X$.

Let $X$ be a Stein space without isolated points and $G$ a subsemigroup of $\text{Pic}(X)$ such that $1_X \in G$. Let $K$ be a compact set of $X$. Then the set $K_{X,G}$ is compact (see Proposition 4.1 and Corollary 4.5) and we have that $hK_X \subset K_{X,G} \subset K_X$.

3. Lemmas

Lemma 3.1. Let $X$ be a Stein space and let $L \in \text{Pic}(X)$. Let $s \in \Gamma(X, \mathcal{O}(L))$ and assume that the set $A := \{s = 0\}$ is nowhere dense in $X$. Then for every $\varphi \in \mathcal{O}(X \setminus A)$, for every compact set $K$ of $X \setminus A$, and for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $u \in \Gamma(X, \mathcal{O}(L^N))$ such that $\|\varphi - (u/s^N)\|_K < \varepsilon$.

Proof. Let $\{a_{ij}\} \subset Z^1(\{U_i\}_{i \in I}, \mathcal{O}^*)$ be a system of transition functions of $L$. Let $s \cong \{s_i\}$ be such that $s_i = a_{ij}s_j$ on $U_i \cap U_j$ for every $i, j \in I$. The set $\{s_i = 0\}$ is nowhere dense in $U_i$ for every $i \in I$. Then the subsheaf $\mathcal{F}$ of $\mathcal{M}$ on $X$ is defined by $\mathcal{F} = (1/s_i) \cdot \mathcal{O}$ on $U_i$ for every $i \in I$. Take an arbitrary $\varphi \in \mathcal{O}(X \setminus A)$, an arbitrary compact set $K$ of $X \setminus A$, and an arbitrary $\varepsilon > 0$. Take an analytic polyhedron $W$ of $X$ such that $K \subset W$, where $W$ is of the form

$$W = G \cap \{ |f_1| < 1, |f_2| < 1, \ldots, |f_m| < 1 \} \subset G,$$

$G$ is an open set of $X$ and $f_1, f_2, \ldots, f_m \in \mathcal{O}(X)$. Then the induced map

$$(f_1, f_2, \ldots, f_m) : W \to \Delta^m$$

is proper, where $\Delta := \{ t \in \mathbb{C} \mid |t| < 1 \}$.
Kaup-Kaup [13, p. 226]). Since \( \mathcal{F} \) is coherent (see Grauert-Remmert [9, p. 119]), there exist finitely many sections \( h_1, h_2, \ldots, h_n \in \Gamma(X, \mathcal{F}) \) such that the germs \( (h_1)_x, (h_2)_x, \ldots, (h_n)_x \) generate \( \mathcal{F}_x \) as an \( \mathcal{O}_x \)-module for every \( x \in W \) by the Cartan theorem A. Let \( t_{iv} := s_i h_{iv} \in \mathcal{O}(U_i) \) for every \( i \in I \) and for every \( \nu = 1, 2, \ldots, n \). Take an arbitrary \( x \in W \cap A \). Take an index \( i \in I \) such that \( x \in U_i \). Since \( 1/(s_i)_x \in \mathcal{F}_x \), there exist \( (g_1)_x, (g_2)_x, \ldots, (g_n)_x \in \mathcal{O}_x \) such that \( 1/(s_i)_x = \sum_{\nu=1}^{n} (g_{\nu})_x (h_{\nu})_x \). Since we have that \( \sum_{\nu=1}^{n} g_{\nu} t_{iv} = 1 \) near \( x \), there exists \( \nu_0 \in \{1, 2, \ldots, n\} \) such that \( t_{iv_0}(x) \neq 0 \). On the other hand \( s_i(x) = 0 \). Therefore

\[
\lim_{y \to x} \frac{|h_{\nu_0}(y)|}{|s_i(y)|} = +\infty.
\]

It follows that the induced map \( (f_1, f_2, \ldots, f_m, h_1, h_2, \ldots, h_n) : W \setminus A \to \Delta^m \times \mathbb{C}^n \) is proper. There exist \( l \in \mathbb{N} \) and a holomorphic map \( \theta = (\theta_1, \theta_2, \ldots, \theta_l) : X \to \mathbb{C}^l \) such that \( \theta \) is injective and regular\(^1\) on \( W \) (see Kaup-Kaup [13, p. 233]). Let

\[
\eta := (\theta_1, \theta_2, \ldots, \theta_l, f_1, f_2, \ldots, f_m, h_1, h_2, \ldots, h_n) : X \setminus A \to \mathbb{C}^{l+m+n}.
\]

Then the induced map \( \eta : W \setminus A \to \mathbb{C}^l \times \Delta^m \times \mathbb{C}^n \) is injective, regular, and proper on \( W \setminus A \). Therefore \( B := \eta(W \setminus A) \) is an analytic set of \( \mathbb{C}^l \times \Delta^m \times \mathbb{C}^n \) and the induced map \( \eta : W \setminus A \to B \) is biholomorphic. Since the function \( \varphi \circ \eta^{-1} \) is holomorphic on \( B \), there exists a function \( \alpha \in \mathcal{O}(\mathbb{C}^l \times \Delta^m \times \mathbb{C}^n) \) such that \( \alpha|_B = \varphi \circ \eta^{-1} \). Since \( \varphi(K) \) is a compact set of \( \mathbb{C}^l \times \Delta^m \times \mathbb{C}^n \), there exists a polynomial function \( \beta \) on \( \mathbb{C}^{l+m+n} \) such that \( \|\alpha - \beta\|_{\varphi(K)} < \varepsilon \). Since \( \beta \circ \eta \) is a polynomial of \( \theta_1, \theta_2, \ldots, \theta_l, f_1, f_2, \ldots, f_m, h_1, h_2, \ldots, h_n \), there exists \( N \in \mathbb{N} \) and a polynomial \( u_i \) of \( \theta_1, \theta_2, \ldots, \theta_l, f_1, f_2, \ldots, f_m, t_{i_1}, t_{i_2}, \ldots, t_{i_n}, s_i \) such that \( \beta \circ \eta = u_i/s_i^N \) on \( U_i \) for every \( i \in I \) and the number \( N \) does not depend on the choice of the index \( i \). Then we have that \( u_i = (s_i/s_j)^N u_j = a_{ij}^N u_j \) on \( U_i \cap U_j \) for every \( i, j \in I \). It follows that \( u \cong \{u_i\} \in \Gamma(X, L^N) \) and we have that \( \|\varphi - (u/s^N)\|_K < \varepsilon \). \( \square \)

\(^1\) A holomorphic map \( \varphi : X \to Y \) between complex spaces \( X \) and \( Y \) is said to be regular at \( x \in X \) if the map \( (d\varphi)_x : m_\varphi(x)/m^2_\varphi(x) \to m_\varphi(x)/m^2_\varphi(x), h_{\varphi(x)} + m^2_\varphi(x) \mapsto (h \circ \varphi)_x + m^2_\varphi(x) \), is surjective (see Grauert [7, p. 333]).
**Lemma 3.2.** Let $X$ be a Stein space. Let $L \in \text{Pic}(X)$ and $s \in \Gamma(X, \mathcal{O}(L))$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of irreducible components of $X$, $\Lambda' := \{ \lambda \in \Lambda \mid s \not\equiv 0 \text{ on } X_\lambda \}$, and $X' := \bigcup_{\lambda \in \Lambda'} X_\lambda$. Then there exists $t \in \Gamma(X, \mathcal{O}(L))$ such that $t|_{X'} = s|_{X'}$ and the set $\{ t = 0 \}$ is nowhere dense in $X$.

**Proof.** Let $\Lambda'' := \Lambda \setminus \Lambda'$. Take a point $p_\lambda \in X_\lambda \setminus \left( \bigcup_{\mu \not\equiv \lambda} X_\mu \right)$ for every $\lambda \in \Lambda''$. Then $P := \{ p_\lambda \mid \lambda \in \Lambda'' \}$ and $S := X' \cup P$ are analytic sets in $X$. By the Cartan theorem B the restriction map

$$\Gamma(X, \mathcal{O}(L)) \rightarrow \Gamma(X, \mathcal{O}(L)/i(S) \cdot \mathcal{O}(L)) \cong \Gamma(X', \mathcal{O}(L|_{X'})) \oplus \mathbb{C}^P$$

is surjective, where $i(S)$ denotes the maximal defining ideal sheaf of $S$. Therefore there exists $t \in \Gamma(X, \mathcal{O}(L))$ such that $t|_{X'} = s|_{X'}$ and $t(p_\lambda) \neq 0$ for every $\lambda \in \Lambda''$. Then the set $\{ t = 0 \}$ is nowhere dense in $X$. □

**Lemma 3.3** (cf. Grauert-Remmert [8, p. 129]) Let $X$ be a Stein space. Let $L \in \text{Pic}(X)$ and $s \in \Gamma(X, \mathcal{O}(L))$. Then the open set $X \setminus \{ s = 0 \}$ is Stein.

**Proof.** Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of irreducible components of $X$, $\Lambda' := \{ \lambda \in \Lambda \mid s \not\equiv 0 \text{ on } X_\lambda \}$, and $X' := \bigcup_{\lambda \in \Lambda'} X_\lambda$. By Lemma 3.2 there exists $t \in \Gamma(X, \mathcal{O}(L))$ such that $t|_{X'} = s|_{X'}$ and the set $B := \{ t = 0 \}$ is nowhere dense in $X$. By the proof of Lemma 3.1 there exists $h \in \mathcal{O}(X \setminus B)$ such that $\lim_{y \to x, y \in X \setminus B} |h(y)| = +\infty$ for every $x \in B$. It follows that $X \setminus B$ is Stein and therefore $X \setminus \{ s = 0 \} = X' \cap (X \setminus B)$ is also Stein. □

**Lemma 3.4.** Let $X$ be a Stein space and $G$ a subsemigroup of $\text{Pic}(X)$. Then for every compact set $K$ of $X$ we have that

$$K_{X,G} = \bigcap_{A \in \mathcal{Y}(G,K)} K_{X \setminus A}.$$ 

**Proof.** We have that $K_{X,G} = \bigcap_{A \in \mathcal{Y}(G,K)} (X \setminus A) \supset \bigcap_{A \in \mathcal{Y}(G,K)} K_{X \setminus A}$. We prove the other inclusion. Let $x \in K_{X,G}$. Take an arbitrary $A \in \mathcal{Y}(G,K)$. There exist $L \in G$ and $s \in \Gamma(X, \mathcal{O}(L))$ such that $A = \{ s = 0 \}$ and $A \cap K = \emptyset$. Since $x \in K_{X,G}$, we have that $x \in X \setminus A$. 

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Assume that \( x \notin \tilde{K}_{X \setminus A} \). Then there exists \( \varphi \in \mathcal{O}(X \setminus A) \) such that \( \varphi(x) \neq \varphi(K) \). Let \( \{X_\lambda\}_{\lambda \in \Lambda} \) be the set of irreducible components of \( X \), \( \Lambda' := \{ \lambda \in \Lambda \mid s \neq 0 \text{ on } X_\lambda \} \), and \( X' := \bigcup_{\lambda \in \Lambda'} X_\lambda \). By Lemma 3.2 there exists \( t \in \Gamma(X, \mathcal{O}(L)) \) such that \( t|_{X'} = s|_{X'} \) and the set \( B := \{ t = 0 \} \) is nowhere dense in \( X \). By Lemma 3.3 the open set \( X \setminus B \) is Stein. Since \( X \setminus A = X' \cap (X \setminus B) \) is an analytic set of \( X \setminus B \), there exists \( \Psi \in \mathcal{O}(X \setminus B) \) such that \( \Psi|_{X \setminus A} = \varphi - \varphi(x) \). By Lemma 3.1 there exist \( N \in \mathbb{N} \) and \( u \in \Gamma(X, \mathcal{O}(L^N)) \) such that \( \|\Psi - (u/t^N)\|_{K \cup \{x\}} < \delta/2 \), where \( \delta := \min_{y \in K} |\Psi(y)| = \min_{y \in K} |\varphi(y) - \varphi(x)| > 0 \). Let \( h := u/t^N \). Since 
\[
|h(y)| \geq |\Psi(y)| - |\Psi(y) - h(y)| > \delta - \delta/2 = \delta/2 > |h(x)|
\]
for every \( y \in K \), we have that \( h - h(x) \neq 0 \text{ on } K \). Therefore \( \{u' = 0\} \cap K = \emptyset \), where \( u' := u - h(x)t^N \). Since \( L^N \in G \) and \( u' \in \Gamma(X, \mathcal{O}(L^N)) \), and \( x \in \{u' = 0\} \), we have that \( x \notin \tilde{K}_{X,G} \), which is a contradiction. Thus we proved that \( \tilde{K}_{X,G} \subset \bigcap_{\lambda \in \Lambda} \tilde{K}_{X \setminus \lambda} \).

4. Compactness of the meromorphically convex hull

We have the following proposition on the compactness of the meromorphically convex hull of a compact set of a Stein space.

**Proposition 4.1.** Let \( X \) be a Stein space, \( G \) a subsemigroup of \( \text{Pic}(X) \), and \( K \) a compact set of \( X \). Then we have the following two statements.

i) If \( \mathcal{S}(G, K) \neq \emptyset \), then the set \( \tilde{K}_{X,G} \) is compact.

ii) If \( \mathcal{S}(G, K) = \emptyset \), then \( \tilde{K}_{X,G} = X \).

**Proof.** i). If \( A \in \mathcal{S}(G, K) \), then \( X \setminus A \) is Stein by Lemma 3.3 and therefore \( \tilde{K}_{X \setminus A} \) is compact. It follows that we have the assertion by Lemma 3.4. ii). Clear. □

There can exist a Stein space \( X \), a subsemigroup \( G \) of \( \text{Pic}(X) \), and a compact set \( K \) of \( X \) such that \( \mathcal{S}(G, K) = \emptyset \). We have the following example of Stein [23, p. 743] (see also Hirschowitz [11, p. 48]). We denote by \( i \) the imaginary unit.
Example 4.2. Let

\[ X := (\mathbb{C}^*)^2 \subset \mathbb{C}^2, \quad A := \{(z, w) \in X \mid z = w^1\}, \quad \text{and} \]

\[ K := \{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}. \]

Let \( L \) be the holomorphic line bundle on \( X \) associated to the divisor \( A \). Then we have that \( \mathcal{H}(L, K) = 0 \).

**Proof.** We have that \( \langle A, K \rangle = 1 \), where the angle brackets denote the intersection number in \( \mathbb{Z} \). Assume that \( \mathcal{H}(L, K) \neq 0 \). Then there exist \( N \in \mathbb{N} \) and \( s \in \Gamma(X, \mathcal{O}(L^N)) \) such that \( \{s = 0\} \cap K = \emptyset \). Since \( L^N \) is holomorphically trivial on \( X \setminus \{s = 0\} \), we have that \( N \langle A, K \rangle = \langle NA, K \rangle = 0 \). It is a contradiction. □

We also have the following example of Oka [19] (see also Nishino [17, p. 92]), the proof of which is the same as above.

Example 4.3. Let

\[ X := \{(z, w) \in \mathbb{C}^2 \mid \frac{2}{3} < |z| < 1, \quad \frac{2}{3} < |w| < 1\} \subset \mathbb{C}^2, \]

\[ A := \{(z, w) \in X \mid \Im(z) \geq 0, \quad w - z + 1 = 0\}, \quad \text{and} \]

\[ K := \{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = \frac{5}{6}\}. \]

Let \( L \) be the holomorphic line bundle on \( X \) associated to the divisor \( A \). Then we have that \( \mathcal{H}(L, K) = 0 \).

We have the following characterization for a subsemigroup \( G \) of \( \text{Pic}(X) \) such that \( \mathcal{H}(G, K) \neq \emptyset \) for every compact set \( K \) of \( X \).

**Proposition 4.4.** Let \( X \) be a Stein space and \( G \) a subsemigroup of \( \text{Pic}(X) \). Then the following two conditions are equivalent.

1. For every compact set \( K \) of \( X \) we have that \( \mathcal{H}(G, K) \neq \emptyset \).

2. For every relatively compact open set \( D \) of \( X \) there exists \( L \in G \) such that \( L|_D = 1_D \).

\(^2\) For the definition of the intersection number see for example Hamano [10, Definition 2.3].
Proof. (1) \implies (2). Let $D$ be an arbitrary relatively compact open set of $X$. By assumption there exist $L \in G$ and $s \in \Gamma(X, \mathcal{O}(L))$ such that $\{s = 0\} \cap \overline{D} = \emptyset$. Since $s \neq 0$ on $D$, the bundle $L$ is holomorphically trivial on $D$.

(2) \implies (1). Let $K$ be an arbitrary compact set of $X$. Take an $\mathcal{O}(X)$-convex Stein open set $D$ of $X$ such that $K \subset D \subset X$. Since $\tilde{K}_D$ is compact and $D$ is not compact, there exists a point $p \in D \setminus K_D$ and there exists $f \in \mathcal{O}(D)$ such that $f(p) \notin f(K)$. Then $\delta := \min_{z \in K} |f(z) - f(p)| > 0$. By assumption there exists $L \in G$ such that $L$ is holomorphically trivial on $D$. Since the image of the restriction map $\Gamma(X, \mathcal{O}(L)) \to \Gamma(D, \mathcal{O}(L)) \cong \mathcal{O}(D)$ is dense in $\mathcal{O}(D)$ (see Markoe [15, Lemma 1.3]), there exists $s \in \Gamma(X, \mathcal{O}(L))$ such that $\|f - f(p) - s|_D\|_K < \delta/2$. Then we have that $A \cap K = \emptyset$, where $A := \{s = 0\}$. It follows that $A \in \mathcal{S}(G, K)$ and therefore $\mathcal{S}(G, K) \neq \emptyset$. \hfill \Box

Corollary 4.5. Let $X$ be a Stein space and $G$ a subsemigroup of Pic($X$). If $1_X \in G$, then for every compact set $K$ of $X$ we have that $\mathcal{S}(G, K) \neq \emptyset$.

The converse of Corollary 4.5 is not true. There exists a Stein manifold $X$ such that $\text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) = 0$ and $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$ contains a nontorsion element $L$ (see Coltoiu [5, Examples 1–4]). By Coltoiu [5, Theorem 1] the subsemigroup $G := \{L^\nu \mid \nu \in \mathbb{N}\}$ of Pic($X$) satisfies condition (2) in Proposition 4.4. Then $\mathcal{S}(L, K) \neq \emptyset$ for every compact set $K$ of $X$ whereas $1_X \notin G$.

5. Theorem and corollaries

We have the following meromorphic approximation theorem in a Stein space with respect to a subsemigroup of the Picard group.

Theorem 5.1. Let $X$ be a Stein space and $G$ a subsemigroup of Pic($X$). Let $K$ be a compact set of $X$ such that $\tilde{K}_{X,G} = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist $L \in G$ and $f, g \in \Gamma(X, \mathcal{O}(L))$ such that the set $\{g = 0\}$ is nowhere dense in $X$, $g \neq 0$ on $K$, and $\|\varphi - (f/g)\|_K < \varepsilon$.

Proof. We use the method of the proof of Hirschowitz [11, Théorème 2]. Take an open set $D$ of $X$ such that $\varphi \in \mathcal{O}(D)$ and $K \subset D$. Since the
holomorphically convex hull $\hat{K}_X$ of $K$ in $X$ is compact, the set $M := \hat{K}_X \setminus D$ is also compact. Since

$$
\bigcap_{A \in \mathcal{U}(G, K)} \hat{K}_X \setminus A \subset \bigcap_{A \in \mathcal{U}(G, K)} (X \setminus A) = \hat{K}_X \setminus G = K \subset D,
$$

we have that $M \cap \left( \bigcap_{A \in \mathcal{U}(G, K)} \hat{K}_X \setminus A \right) = \emptyset$. If $A \in \mathcal{S}(G, K)$, then by Lemma 3.3 the open set $X \setminus A$ is Stein and therefore the set $\hat{K}_X \setminus A$ is compact. It follows that there exist finitely many $A_1, A_2, \ldots, A_m \in \mathcal{S}(G, K)$ such that $M \cap \left( \bigcap_{\mu=1}^m \hat{K}_X \setminus A_\mu \right) = \emptyset$. For every $\mu = 1, 2, \ldots, m$ there exist $L_\mu \in G$ and $s_\mu \in \Gamma(X, \mathcal{O}(L_\mu))$ such that $A_\mu = \{ s_\mu = 0 \}$ and $K \cap A_\mu = \emptyset$. Then we have that $L := L_1 \otimes L_2 \otimes \cdots \otimes L_m \in G$, $s := s_1 \otimes s_2 \otimes \cdots \otimes s_m \in \Gamma(X, \mathcal{O}(L))$, $A := \{ s = 0 \} = \bigcup_{\mu=1}^m A_\mu$, $K \cap A = \emptyset$, and

$$
\hat{K}_X \setminus A \subset \left( \bigcap_{\mu=1}^m \hat{K}_X \setminus A_\mu \right) \cap \hat{K}_X \subset (X \setminus M) \cap \hat{K}_X \subset D.
$$

By the holomorphic approximation theorem of Weil-Oka (see Grauert [6, Satz 6] or Kaup-Kaup [13, p. 296]) there exists $\psi \in \mathcal{O}(X \setminus A)$ such that $\| \psi - \hat{\psi} \|_{\hat{K}_X \setminus A} < \epsilon/2$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of irreducible components of $X$, $\Lambda' := \{ \lambda \in \Lambda \mid s \neq 0 \text{ on } X_\lambda \}$, and $X' := \bigcup_{\lambda \in \Lambda'} X_\lambda$. By Lemma 3.2 there exists $t \in \Gamma(X, \mathcal{O}(L))$ such that $t|_{X'} = s|_{X'}$ and the set $B := \{ t = 0 \}$ is nowhere dense in $X$. Since $X \setminus B$ is Stein by Lemma 3.3 and $X \setminus A = X' \cap (X \setminus B)$ is an analytic set of $X \setminus B$, we have that $\hat{K}_{X \setminus A} = \hat{K}_{X \setminus B}$ and there exists $\Psi \in \mathcal{O}(X \setminus B)$ such that $\Psi|_{X \setminus A} = \psi$. By Lemma 3.1 there exist $N \in \mathbb{N}$ and $u \in \Gamma(X, \mathcal{O}(L^N))$ such that $\| \Psi - (u/t^N) \|_K < \epsilon/2$. It follows that

$$
\| \varphi - (u/t^N) \|_K \leq \| \varphi - \psi \|_K + \| \Psi - (u/t^N) \|_K
= \| \varphi - \psi \|_{\hat{K}_X \setminus A} + \| \Psi - (u/t^N) \|_K
< \epsilon/2 + \epsilon/2 = \epsilon
$$

and we have that $L^N \in G$, $t^N \in \Gamma(X, \mathcal{O}(L^N))$, $\{ t^N = 0 \} = B$, and $t^N \neq 0$ on $K$. □
Corollary 5.2. Let $X$ be a Stein space, $D$ an open set of $X$, and $K$ a compact set of $D$. Let $G$ be a subsemigroup of $\text{Pic}(X)$ such that $\mathcal{S}(G, K) \neq \emptyset$. Then the following two conditions are equivalent.

1. The set $\tilde{K}_{X,G} \cap D$ is compact.

2. $\tilde{K}_{X,G} \subset D$.

Proof. (1) $\rightarrow$ (2). Let $M := \tilde{K}_{X,G}$, $M_1 := M \cap D$, and $M_2 := M \cap (X \setminus D)$. By assumption the set $M_1$ is compact. Since $M$ is compact by Proposition 4.1, the set $M_2$ is also compact. Therefore there exist open sets $U_1$ and $U_2$ of $X$ such that $M_1 \subset U_1$, $M_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$. Let $U := U_1 \cup U_2$. Let $\varphi \in \mathcal{O}(U)$ be the function defined by $\varphi = \nu$ on $U_{\nu}$ for each $\nu = 1, 2$. Since $\tilde{M}_{X,G} = M$ by Proposition 2.1, there exist $L \in G$ and $s, t \in \Gamma(X, \mathcal{O}(L))$ such that $\{t = 0\}$ is nowhere dense in $X$, $t \neq 0$ on $M$, and $\|\varphi - (s/t)\|_M < 1/2$ by Theorem 5.1. Assume that $M_2 \neq \emptyset$ and take a point $p \in M_2$. Let $h := s/t \in \mathcal{M}(X)$ and $u := s - h(p)t \in \Gamma(X, \mathcal{O}(L))$. Since

$$
\|h - h(p)\|_{M_1} = \|1 - (1 - h) + (2 - h(p))\|_{M_1}
\geq 1 - \|1 - h\|_{M_1} - |2 - h(p)|
\geq 1 - 2\|\varphi - h\|_M > 0,
$$

we have that $h - h(p) \neq 0$ on $M_1$. It follows that $u \neq 0$ on $K$. Since $u(p) = 0$, we have that $p \notin M$. It is a contradiction. It follows that $M_2 = \emptyset$ and therefore $M \subset D$.

(2) $\rightarrow$ (1). The assertion is clear because the set $\tilde{K}_{X,G}$ is compact by Proposition 4.1.

The following Corollary 5.3 generalizes Hirschowitz [11, Théorème 2].

Corollary 5.3. Let $X$ be a Stein space without isolated points. Let $K$ be a compact set of $X$ such that $hK_X = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exists $h \in \mathcal{M}(X) \cap \mathcal{O}(K)$ such that $\|\varphi - h\|_K < \varepsilon$.

Proof. By Proposition 2.2 we have that $hK_X = \tilde{K}_{X,\text{Pic}(X)}$. Then we have the assertion by Theorem 5.1.
Corollary 5.4. Let $X$ be a Stein space and let $L \in \text{Pic}(X)$. Let $K$ be a compact set of $X$ such that $K_X, L = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $f, g \in \Gamma(X, \mathcal{O}(L^N))$ such that the set \{\varphi = 0\} is nowhere dense in $X$, $g \neq 0$ on $K$, and $\|\varphi - (f/g)\|_{L^N} < \varepsilon$.

Proof. We have the assertion by Theorem 5.1.

Corollary 5.5. Let $X$ be a Stein space. Let $K$ be a compact set of $X$ such that $K_X = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist $f, g \in \mathcal{O}(X)$ such that the set \{\varphi = 0\} is nowhere dense in $X$, $g \neq 0$ on $K$, and $\|\varphi - (f/g)\|_{L^N} < \varepsilon$.

Proof. We have the assertion by Corollary 5.4.

Thus we obtained a different proof of Abe [1, Theorem 11]. We also have the following Corollary 5.6 which generalizes Nguyen [16, Lemma 2.2].

Corollary 5.6. Let $S$ be a Stein manifold and $X$ an open set of $S$. Let $K$ be a compact set of $X$ such that $K_X = K$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exist $f, g \in \mathcal{O}(X)$ such that $g \neq 0$ on $K$ and $\|\varphi - (f/g)\|_{L^N} < \varepsilon$.

Proof. We first consider the case when $X$ is connected. Let $\pi' : X' \to S$, $\lambda : X \to X'$ be the envelope of holomorphy of the univalent domain $i : X \to Z$. Let $M := \lambda(K)$. Let $x' \in \tilde{M}_X \cap \lambda(X)$. There exists $x \in X$ such that $\lambda(x) = x'$. Then we have that $x \in \tilde{K}_X = K$ because for every $f \in \mathcal{O}(X)$ there exists $f' \in \mathcal{O}(X')$ such that $f = f' \circ \lambda$ and we have that $f(x) = f'(x') \in f'(M) = f(K)$. It follows that $x' \in M$ and therefore $\tilde{M}_X \cap \lambda(X) \subset M$. Conversely it is clear that $M \subset \tilde{M}_X \cap \lambda(X)$. It follows that $\tilde{M}_X \cap \lambda(X) = M$. Since $X'$ is Stein (see Adachi et al. [4, Lemma 3]) and $\lambda(X)$ is an open set of $X'$, we have that $\tilde{M}_X = M$ by Corollary 5.2. Since $\lambda : X \to \lambda(X)$ is biholomorphic, the function $\varphi \circ \lambda^{-1}$ is holomorphic near $M$. By Corollary 5.5 there exist $f', g' \in \mathcal{O}(X')$ such that $g' \neq 0$ on $M$ and $\|\varphi \circ \lambda^{-1} - (f'/g')\|_{L^N} < \varepsilon$. Let $f := f' \circ \lambda$ and $g := g' \circ \lambda$. Then we have that $g \neq 0$ on $K$ and $\|\varphi - (f/g)\|_{L^N} < \varepsilon$. Next we consider the general case. Let $Z$ be an arbitrary connected component of $X$. Let $L := K \cap Z$. Then
we have that $\tilde{L}_Z = \tilde{L}_X \cap Z \subset \tilde{K}_X \cap Z = K \cap Z = L$ and therefore $\tilde{L}_Z = L$. Thus the assertion is reduced to the case when $X$ is connected.

We denote by $\mathbb{C}[z_1, z_2, \ldots, z_n]$ the algebra of polynomial functions on $\mathbb{C}^n$. A compact set $K$ of $\mathbb{C}^n$ is said to be rationally convex in $\mathbb{C}^n$ if $\tilde{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]} = K$, where

$$\tilde{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]} := \{x \in \mathbb{C}^n \mid f(x) \in f(K) \text{ for every } f \in \mathbb{C}[z_1, z_2, \ldots, z_n]\}$$

is the rationally convex hull of $K$ in $\mathbb{C}^n$. We have the following rational approximation theorem of Weil-Oka (see Oka [18, p. 254] and Stolzenberg [24, p. 283]).

**Corollary 5.7.** Let $K$ be a rationally convex compact set of $\mathbb{C}^n$. Then for every $\varphi \in \mathcal{O}(K)$ and for every $\varepsilon > 0$ there exists a rational function $h$ on $\mathbb{C}^n$ holomorphic in a neighborhood of $K$ such that $\|\varphi - h\|_K < \varepsilon$.

**Proof.** Since $\tilde{K}_{\mathbb{C}^n} = \tilde{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]} = K$ (see Abe [1, p. 265]), there exist $f, g \in \mathcal{O}(\mathbb{C}^n)$ such that $g \neq 0$ on $K$ and $\|\varphi - (f/g)\|_K < \varepsilon/2$ by Corollary 5.5. Since $\mathbb{C}[z_1, z_2, \ldots, z_n]$ is dense in $\mathcal{O}(\mathbb{C}^n)$, there exist $\lambda, \mu \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ such that $\|f - \lambda\|_K < \varepsilon/(4C)$ and $\|g - \mu\|_K < \min\{\varepsilon/(4C), \min_K |g|/2\}$, where $C := 2 \max\{\|f\|_K, \|g\|_K\}/(\min_K |g|)^2$. Then we have that $|\mu| > \min_K |g|/2 > 0$ on $K$ and $\|f/g\) - (\lambda/\mu)\|_K < \varepsilon/2$. It follows that the rational function $h := \lambda/\mu$ is holomorphic near $K$ and we have that $\|\varphi - h\|_K < \varepsilon$.

Since every compact set $K$ of $\mathbb{C}$ is rationally convex in $\mathbb{C}$, the rational approximation theorem of Weil-Oka above generalizes the weak rational approximation theorem of Runge in $\mathbb{C}$ (see Rudin [22, Theorem 13.6]). On the other hand see Abe [2] for a trial to generalize the strong rational approximation theorem (see Rudin [22, Theorem 13.9]).

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3 Oka [18] is usually understood as the paper on the Cousin-I problem on polynomial polyhedra and the polynomial approximation in $\mathbb{C}^n$ because the original proof contains the argument valid only to polynomial polyhedra, which is noted in the errata in Oka [20, p. 521] (see also the footnote *) at the bottom of Oka [21, p. 4]).
References


Makoto Abe
Faculty of Life Sciences
Kumamoto University
Kumamoto 862-0976, JAPAN
e-mail: mabe@kumamoto-u.ac.jp

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