<table>
<thead>
<tr>
<th>Title</th>
<th>Continuous flattening of platonic polyhedra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Itoh, Jin-ichi; Nara, Chie</td>
</tr>
<tr>
<td>Citation</td>
<td>Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and ...)</td>
</tr>
<tr>
<td>Issue date</td>
<td>2011</td>
</tr>
<tr>
<td>Type</td>
<td>Journal Article</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2298/26242">http://hdl.handle.net/2298/26242</a></td>
</tr>
<tr>
<td>Right</td>
<td>© 2011 Springer-Verlag</td>
</tr>
</tbody>
</table>

Kumamoto University
Continuous Flattening of Platonic Polyhedra

Jin-ichi Itoh$^1$ and Chie Nara$^2$

$^1$ Faculty of Education, Kumamoto University, Kumamoto, 860-8555, Japan. j-itoh@kumamoto-u.ac.jp

$^2$ Liberal Arts Education Center, Aso Campus, Tokai University, Aso, Kumamoto, 869-1404, Japan. cnara@ktmail.tokai-u.jp

Abstract. We prove that each Platonic polyhedron $P$ can be folded into a flat multilayered face of $P$ by a continuous folding process for polyhedra.

1 Introduction

An empty juice box made of paper may be flattened in a manner illustrated in Fig. 1: By pushing in the four sides of box (with dashed lines), the front and the back of the box pop out and the whole box squashes flat. This is an example of a folded polyhedron. We use the terminology polyhedron for a polyhedral surface which is permitted to touch itself but not self-intersect. A crease is a line segment on a piece of paper. A crease pattern is a collection of creases drawn on paper, meeting only at common end points. A flat folding of a polyhedron is a folding by creases into a multilayered but planar shape. We mainly adopt the terminologies in [4].

In this paper we prove that for each Platonic polyhedron $P$ (the cube, the regular tetrahedron, the regular octahedron, the regular dodecahedron, or the regular icosahedron) there is a flat folding of $P$ onto its original face by a continuous folding process for polyhedra. This relates to the following problem, proposed in [4] (p. 279) by E. Demaine and J. O’Rourke:

Open Problem. Can every folded state of polyhedral pieces of paper be reached by a continuous folding process?

We show in Theorem 2 and Remark 3 that the juice box in Fig. 1 is flattened to a multilayered face by a continuous folding process. We begin with the following definition.

Definition 1. Let $P$ be a polyhedron in the Euclidean space $\mathbb{R}^3$. We say $P$ is flattened by a continuous folding process for polyhedra on its face, if there is a continuous family of polyhedra $\{P_t : 0 \leq t \leq 1\}$ satisfying the following conditions:

1. for each $0 \leq t \leq 1$, $P_t$ is combinatorially isomorphic to the polyhedron $P'_t$ which is obtained from $P$ by subdividing some faces (so some faces may be

* This work was supported by KAKENHI(23540160).
included in the same face of $P$ but $P'_t$ is congruent to $P$) such that the corresponding faces of $P'_t$ and $P_t$ are congruent,

(2) $P_0 = P$, and

(3) $P_1$ is a flat polyhedron which is a multilayered face for some face of $P$.
We call $P_1$ a flat folded polyhedron of $P$ on its face.

We introduce three theorems which imply properties of creases used to flatten a polyhedron by a continuous folding process.

Cauchy’s Theorem [1]. Any convex polyhedron is rigid. Precisely, if two convex polyhedra $P, P'$ are combinatorially equivalent and their corresponding faces are congruent, then $P$ and $P'$ are congruent.

Does Cauchy’s theorem hold without the hypothesis of convexity? R. Connelly gave a negative answer by a counterexample.

Connelly’s Theorem [2,3]. There is a flexible polyhedron. Precisely, there is a continuous family of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) such that for every \( t \neq 0 \), the corresponding faces of $P_0$ and $P_t$ are congruent while polyhedra $P_0$ and $P_t$ are not congruent.

Connelly’s counterexample is flexible, while its volume is invariant. Is there a flexible polyhedron which changes its volume?

Sabitov’s Theorem [6, 7]. The volume of any polyhedron is invariant under flexing. Precisely, if there is a continuous family of polyhedra \( \{P_t : 0 \leq t \leq 1\} \) such that, for every $t$, the corresponding faces of $P_0$ and $P_t$ are congruent, then the volumes $P_0$ and $P_t$ are equal for all $0 \leq t \leq 1$.

Remark 1. Sabitov’s theorem implies that if a polyhedron $P$ is flattened by a continuous folding process for polyhedra \( \{P_t : 0 \leq t \leq 1\} \), then the crease pattern in $P$ for \( \{P_t : 0 \leq t \leq 1\} \) is not a finite set of line segments.
In Sect. 2 we prove a lemma which plays a key role throughout this paper, followed by three propositions. In Sect. 3, for each Platonic polyhedron $P$ we prove that there is a continuous folding process for polyhedra to obtain a flat folded polyhedron of $P$ on its original face.

This research was motivated by Y. Isokawa’s result [5] on a continuous flattening of the cube.

2 Lemma and Propositions

For $x, y$ in $\mathbb{R}^3$ with the origin $O$, we denote by $xy$ the line segment with endpoints $x, y$ and by $|xy|$ its length (or the distance of $x$ and $y$).

**Definition 2.** Let $Q$ be a connected subset of a polyhedron (made of paper) whose boundary consists of finite line segments. If a set $S$ is obtained by folding $Q$ with finite creases (where $S$ may touch but not intersect itself), then there is a mapping $f$ from $Q$ onto $S$ which is isometric on each closed subregion of $Q$ divided by those creases, original edges and the boundary of $Q$, and such mapping is unique. We call $f$ the piecewise isometric mapping from $Q$ to $S$.

**Definition 3.** Let $R = abcd$ be a rhombus in $\mathbb{R}^3$. Let $o$ be the center of $R$, and $q, q' \in ac$ be two points (possibly, $q = q'$) symmetric with respect to $o$, as shown in Fig. 2(1). Let $S$ be a folded state of $R$ with the crease pattern \{ao, bo, do, oq, qc, bq, dq\} such that its piecewise isometric mapping $f$ from $R$ to $S$ satisfies $f(q) = f(q')$, as indicated in Fig. 2(2)-4. Then we call $S$ a folded rhombus of $R$.

We denote by $(x_1, x_2, x_3)$ the coordinates of a point $x$ in $\mathbb{R}^3$. The following Lemma plays a key role through proofs of the theorems.

**Lemma 1.** Let $R = abcd$ be a rhombus in $\mathbb{R}^3$. For any $l (0 \leq l \leq |ac|)$ and any $m (0 \leq m \leq |bd|)$, there is a folded rhombus $S$ of $R$ such that its piecewise isometric mapping $f$ from $R$ to $S$ satisfies $|f(a)f(c)| = l$ and $|f(b)f(d)| = m$.

**Proof.** Let $R = abcd$ be a rhombus with center at the origin $O$ in $\mathbb{R}^3$. Denote by $\alpha$ the edge length of $R$, that is, $|ab| = |bc| = |cd| = |da| = \alpha$. Let $l$ and $m$ be any numbers satisfying $0 \leq l \leq |ac|$ and $0 \leq m \leq |bd|$. If $l = |ac|$ or $m = |bd|$, a folded rhombus of $R$ which satisfies all conditions in the lemma is obtained by folding $R$ along $ac$ or $bd$ respectively. So let $0 \leq l < |ac|$ and $0 \leq m < |bd|$.

Let $a = (a_1, 0, 0), b = (0, b_2, 0), c = (-a_1, 0, 0)$ and $d = (0, -b_2, 0)$ (Fig. 2(1)). Then $|ab| = \sqrt{a_1^2 + b_2^2} = \alpha$. Define $f(a) = a, f(O) = O, f(b) = (0, m/2, -\sqrt{b_2^2 - (m/2)^2})$ and $f(d) = (0, -m/2, -\sqrt{b_2^2 - (m/2)^2})$. Let $u = (u_1, 0, u_3)$ with $u_3 \geq 0$ be the point satisfying $|uf(b)| = |uf(d)| = \alpha$ and $|uf(a)| = l$, which is uniquely determined by $l < |ac|$ (Fig. 2(2)-4). Define $f(c) = u$. Draw the line $L$ passing through the midpoint of the line segment $f(a)f(c)$ and perpendicular to $f(a)f(c)$. Then $L$ meets the line segment $f(a)O = aO$. Let $q'$ be the
intersection of $L$ and $aO$, and $q$ be the point symmetric to $q'$ about $O$. Define $f(q) = f(q') = q'$.

Since $|f(c)q'| = |aq'|$, then $|f(c)q'| + |q'a| = |aq'| + |q'o| = |ao| = |co|$ holds. Hence the six triangles $\triangle f(a)f(b)O$, $\triangle f(a)f(d)O$, $\triangle f(b)f(q)O$, $\triangle f(d)f(q)O$, $\triangle f(b)f(q)f(c)$ and $\triangle f(d)f(q)f(c)$ are congruent to the original triangles $\triangle abO$, $\triangle adO$, $\triangle bqO$, $\triangle dqO$, $\triangle bqf$ and $\triangle dfc$, respectively. Let $S$ be the union of those six triangles $\triangle f(a)f(b)O$, $\triangle f(a)f(d)O$, $\triangle f(b)f(q)O$, $\triangle f(d)f(q)O$, $\triangle f(b)f(q)f(c)$ and $\triangle f(d)f(q)f(c)$. Then $S$ is the folded rhombus of $R$ which satisfies $|f(a)f(c)| = l$ and $|f(b)f(d)| = m$, and the piecewise isometric mapping from $R$ to $S$ is given as an extension of $f$. \qed

Remark 2. First, the coordinates of $q \in \mathbb{R}^3$ in the lemma are continuous functions with respect to $l$ and $m$.

Second, the lemma holds for folded rhombuses instead of rhombuses. In particular, let $R = abed$ be a folded rhombus in $\mathbb{R}^3$. For any $l \leq l \leq |ac|$ and any $m \leq m \leq |bd|$, there is a folded rhombus $S$ of $R$ and the piecewise isometric mapping $f$ from $R$ to $S$ such that $|f(a)f(c)| = l$ and $|f(b)f(d)| = m$.

In Propositions 1 through 3, we apply the lemma to three specified folded rhombuses which play main roles in obtaining flat folded Platonic polyhedra shown in the section 3.
Proposition 1. Let \( R = abcd \) be a rhombus with center \( o \) such that \( \triangle abd \) and \( \triangle bcd \) are equilateral triangles with circumcenters \( g' \) and \( g \) respectively as in Figure 3(1). Then there is a folded rhombus of \( R \) (denoted by \( S \)) and its piecewise isometric mapping \( f \) such that \(|f(a)f(c)| = |ab|\) and \(|f(b)f(d)| = 0\) as in Fig. 3(2). Moreover, there is a continuous folding process involving folded rhombuses \( \{R_t : 0 \leq t \leq 1\} \) of \( R \) and the piecewise isometric mappings \( f_t \) \((0 \leq t \leq 1)\) from \( R \) to \( R_t \) \((0 \leq t \leq 1)\) such that \( R_0 = R, R_1 = S \), and the crease pattern in \( R \) for \( \{R_t : 0 \leq t \leq 1\} \) is the set \( \{ao, bo, do\} \cup \{op, pc, bp, dp : p \in og\} \) as suggested by dashed lines in Fig. 3(1).

![Fig. 3. A flat folded rhombus](image)

Proof. Let \( R = abcd \) be a rhombus with center \( o \) such that \( \triangle abd \) and \( \triangle bcd \) are equilateral triangles with circumcenters \( g' \) and \( g \) respectively. Apply the lemma to the case \( l = |ab| \) and \( m = 0 \) for \( R \) to obtain a folded rhombus of \( R \), denoted by \( S \), and its piecewise isometric mapping \( f \) such that \(|f(a)f(c)| = |ab|\) and \(|f(b)f(d)| = 0\). By the conditions \(|f(a)f(c)| = |ab|\) and \(|f(b)f(d)| = 0\), the corresponding points to \( q' \) and \( q \) in the lemma are \( g' \) and \( g \), respectively.

Let \( m \) decrease to zero and \( l \) decrease to \(|ab|\) as \( t \) increases from zero to 1. Then by the lemma and Remark 2, there is a continuous process of folded rhombuses \( \{R_t : 0 \leq t \leq 1\} \) of \( R \) such that \( R_0 = R \) and \( R_1 = S \). Denote by \( q_t \) \((0 \leq t \leq 1)\) the point corresponding to \( q \) in the lemma. Then the point \( q_t \) moves from the origin \( O \) to \( g \) as \( t \) increases to 1, and so the crease pattern in \( R \) for \( \{R_t : 0 \leq t \leq 1\} \) is the set \( \{ao, bo, do\} \cup \{op, pc, bp, dp : p \in og\} \). The dashed lines in \( \triangle bgd \) in Fig. 3(1) are creases for some \( P_t \).

\[ \square \]

Proposition 2. Let \( R = abcd \) be a square (rhombus) with center \( o \), as in Figure 4(1). Then there is a folded rhombus of \( R \), denoted by \( S \), and its piecewise isometric mapping \( f \) such that \(|f(a)f(c)| = 0\) and \(|f(b)f(d)| = 0\), as in Figure 4(2). Moreover, there is a continuous folding process involving folded rhombuses.
\{R_t : 0 \leq t \leq 1\} of \( R \) and their piecewise isometric mappings \( \{f_t : 0 \leq t \leq 1\} \) such that \( R_0 = R, R_1 = S \), and the crease pattern in \( R \) for \( \{R_t : 0 \leq t \leq 1\} \) is the set \( \{ao, bo, do\} \cup \{op, pc, bp, dp : p \in oh\} \).

**Proof.** Let \( R = abcd \) be a square (rhombus) with center \( o \) as in Fig. 4(1). Apply the lemma to the case \( l = m = 0 \) for \( R \) to obtain a folded rhombus of \( R \) (denoted by \( S \)) and its piecewise isometric mapping \( f \) such that \( |f(a)f(c)| = 0 \) and \( |f(b)f(d)| = 0 \) as in Fig. 4(2). We can prove Proposition 2 by a process similar to the one used in the proof of Proposition 1, so we omit the details. \( \square \)

**Proposition 3.** Let \( R = abcd \) be a rhombus with center \( o \) which is inscribed in a pair of adjacent congruent regular pentagons (that is, \( |ab|/|bd| = (\sqrt{5}+1)/2 \)) with circumcenters \( h' \) and \( h \), as in Fig. 5(1). Then there is a folded rhombus \( S \) of \( R \) and its piecewise isometric mapping \( f \) such that \( |f(a)f(c)| = |bd| \) and \( |f(b)f(d)| = 0 \), as in Fig. 5(2). Moreover, there is a continuous folding process \( \{R_t : 0 \leq t \leq 1\} \) involving folded rhombuses of \( R \) and their piecewise isometric mappings \( \{f_t : 0 \leq t \leq 1\} \) such that \( R_0 = R, R_1 = S \), and the crease pattern in \( R \) for \( \{R_t : 0 \leq t \leq 1\} \) is the set \( \{ao, bo, do\} \cup \{op, pc, bp, dp : p \in oh\} \) as in Figure 5(1).

**Proof.** Let \( R = abcd \) be a rhombus with center \( o \) which is inscribed in a pair of adjacent congruent regular pentagons (that is, \( |ab|/|bd| = (\sqrt{5}+1)/2 \)), as in Fig. 5(1). Apply the lemma to the case \( l = |bd| \) and \( m = 0 \) for \( R \) to obtain a folded rhombus \( S \) of \( R \) and its piecewise isometric mapping \( f \) such that \( |f(a)f(c)| = |bd| \) and \( |f(b)f(d)| = 0 \), as in Fig. 5(2). We can prove Proposition 3 by a process similar to the one used in the proof of Proposition 1, so we omit the details. \( \square \)
3 Theorems and Proofs

In this section, we show that each Platonic polyhedron can be flattened on its face by a continuous folding process for polyhedra.

**Theorem 1.** The regular tetrahedron is flattened on its face by a continuous folding process for polyhedra.

Precisely, the regular tetrahedron \( P = abcd \) is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in \( P \) is \( \{ao, bo, do\} \cup \{op, pc, bp, dp : p \in og_1\} \), where \( g_1 \) is the circumcenter of \( \triangle bcd \), as shown in Fig. 6(1)(2).

![Fig. 5. A flat folded rhombus inscribed in a regular pentagon](image)

![Fig. 6. A flat folded regular tetrahedron: \( o \) and \( h \) are the midpoints of \( bd \) and \( bc \), respectively, and \( g_1 \) and \( g_2 \) are the circumcenters of \( \triangle bcd \) and \( \triangle abc \), respectively.](image)
Proof. We show that we can flatten a regular tetrahedron \( P \) (made of paper) on its face by pushing in two of its sides and popping out its two remaining faces by a continuous folding process.

Let \( P = abcd \) be the regular tetrahedron in \( \mathbb{R}^3 \), as in Fig. 6(1). Then the two faces \( \triangle abc \) and \( \triangle bcd \) comprise a folded rhombus \( R \). Let \( S \) and \( f \) be a flat folded rhombus of \( R \) and its piecewise isometric mapping such that \( |f(a)f(c)| = |ab| \) and \( |f(b)f(d)| = 0 \), as shown in Fig. 3(2). By Remark 2 and Proposition 1, there is a continuous folding process involving folded rhombuses \( \{ R_t : 0 \leq t \leq 1 \} \) of \( R \) with piecewise linear mappings \( \{ f_t : 0 \leq t \leq 1 \} \) from \( R \) to \( R_t \) such that \( R_0 = R \) and \( R_1 = S \).

Since \( \triangle f_t(a)f_t(b)f_t(c) \) and \( \triangle f_t(a)f_t(d)f_t(c) \) are congruent to the original triangles \( \triangle abc \) and \( \triangle adc \) respectively, we define polyhedra \( \{ P_t : 0 \leq t \leq 1 \} \) by \( P_t = R_t \cup \triangle f_t(a)f_t(b)f_t(c) \cup \triangle f_t(a)f_t(d)f_t(c) \). Then we obtain a continuous folding process of polyhedra \( \{ P_t : 0 \leq t \leq 1 \} \) such that \( P_0 = P \) and \( P_1 \) is a flat folded regular tetrahedron, as shown in Fig. 6(2). Hence for each \( t (0 \leq t \leq 1) \), the piecewise isometric mapping from \( P \) to \( P_t \) is the unique extension of \( f_t \) from \( R \) to \( R_t \). The crease pattern in \( P \) for \( \{ P_t : 0 \leq t \leq 1 \} \) is \( \{ ao, bo, do \} \cup \{ op, pc, bp, dp : p \in oq_1 \} \) where \( g_1 \) is the circumcenter of \( \triangle bcd \).

\[ \square \]

**Theorem 2.** The cube is flattened on its face by a continuous folding process for polyhedra.

Precisely, let \( P \) be the cube with vertices \( v_i (1 \leq i \leq 8) \), \( g_i (1 \leq i \leq 4) \) be the centers of side faces, and \( h_i \) be the midpoints of edges \( v_i v_{i+4} (1 \leq i \leq 4) \) in \( \mathbb{R}^3 \) as shown in Fig. 7(1). Then \( P \) is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in \( P \) is \( \{ v_i h_i, h_i v_{i+4} : 1 \leq i \leq 4 \} \cup \{ h_i p_i, p_i h_{i+1}, v_i p_i, v_{i+4} p_i : p_i \in h_i g_i, 1 \leq i \leq 4 \} \) where \( h_5 \) means \( h_1 \), as shown in Fig. 7(1)(2).

Proof. Let \( P \) be the cube with vertices \( v_i (1 \leq i \leq 8) \) in \( \mathbb{R}^3 \). Let \( g_i (1 \leq i \leq 4) \) be the centers of four side faces \( v_{i+1} v_{i+5} v_{i+4} (i = 1, 2, 3) \) and \( v_4 v_5 v_6 v_8 \), and \( h_i (1 \leq i \leq 4) \) be the midpoints of edges \( v_i v_{i+4} (1 \leq i \leq 4) \), as shown in Fig. 7(1).

Let \( R_t (1 \leq i \leq 4) \) be four folded rhombuses (squares) \( g_1 v_2 g_2 v_6, g_2 v_3 g_3 v_7, g_3 v_4 g_4 v_8 \) and \( g_4 v_1 g_1 v_5 \), which are included in side faces of \( P \). Apply Proposition 2 to those folded squares simultaneously, which means that four vertices \( v_{i+4} (1 \leq i \leq 4) \) move along edges \( v_{i+4} v_i \) to vertices \( v_i (1 \leq i \leq 4) \) at the same speed. By using a similar argument to that in the proof of Theorem 1, we obtain a flat folded cube \( Q \) of \( P \) on its face \( v_1 v_2 v_3 v_4 \) by a continuous folding process for polyhedra \( \{ P_t : 0 \leq t \leq 1 \} \) with piecewise linear mappings \( \{ f_t : 0 \leq t \leq 1 \} \) from \( P \) to \( P_t \) such that \( P_0 = P \) and \( P_1 = Q \), as in Fig. 7(2). The crease pattern in \( P \) for \( \{ P_t : 0 \leq t \leq 1 \} \) is \( \{ v_i h_i, h_i v_{i+4} : 1 \leq i \leq 4 \} \cup \{ h_i p_i, p_i h_{i+1}, v_i p_i, v_{i+4} p_i : p_i \in h_i g_i, 1 \leq i \leq 4 \} \) where \( h_5 \) means \( h_1 \).

\[ \square \]

**Remark 3.** Theorem 2 is extended to a rectangular box \( P \) by considering the shortest edge length of \( P \) as its height. So the juice box in Fig. 1 is flattened on a face of \( P \) by a continuous folding process for polyhedra, which has a different
crease pattern from those in Fig. 1 right, but since we can choose any one of two triangles comprising the rhombus for creases in the lemma, the juice box is flattened on a face of \( P \), as in Fig. 1 right, by a continuous folding process for polyhedra.

**Theorem 3.** The regular octahedron is flattened on its face by a continuous folding process.

Precisely, let \( P \) be the octahedron with vertices \( v_i \) (\( 1 \leq i \leq 6 \)), \( g_i \) be the circumcenter of \( \triangle v_2 v_6 v_5 \) and \( h_1 \) be the midpoint of \( v_2 v_5 \) in \( \mathbb{R}^3 \) as shown in Fig. 8(1). Let \( l \) be the line passing through the circumcenters of \( \triangle v_1 v_2 v_3 \) and \( \triangle v_4 v_5 v_6 \). Let

\[ C = \{v_1 h_1, v_2 h_1, v_5 h_1 \} \cup \{h_1 p, pv_6, v_2 p, v_5 p : p \in h_1 g_1 \} \]

and \( C_T \) be the union of three rotations of \( C \) about \( l \) with radial angles 0, \( 2\pi/3 \) and \( 4\pi/3 \). Then \( P \) is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in \( P \) is \( C_T \).

**Proof.** Let \( P \) be the regular octahedron with vertices \( v_i \) (\( 1 \leq i \leq 8 \)) in \( \mathbb{R}^3 \), and let \( g_i \) (\( 1 \leq i \leq 3 \)) be the circumcenters of three triangular faces (\( \triangle v_2 v_6 v_5 \), \( \triangle v_3 v_4 v_6 \) and \( \triangle v_1 v_5 v_4 \)), as shown in Fig. 8(1).

Let \( R_i \) (\( 1 \leq i \leq 3 \)) be three folded rhombuses \( v_1 v_2 v_6 v_5 \), \( v_2 v_3 v_4 v_6 \) and \( v_3 v_1 v_5 v_4 \) included in \( P \). Denote by \( \tilde{v}_4 \tilde{v}_1 \), \( \tilde{v}_5 \tilde{v}_2 \) and \( \tilde{v}_6 \tilde{v}_3 \) the circular arcs with centers \( v_3 \), \( v_1 \) and \( v_2 \) respectively which are the shorter ones drawn on the planes including the faces \( \triangle v_4 v_3 v_1 \), \( \triangle v_5 v_1 v_2 \) and \( \triangle v_6 v_2 v_3 \) respectively. Apply Proposition 1 to those folded rhombuses simultaneously, which means that three vertices \( v_4 \), \( v_5 \) and \( v_6 \) move along circular arcs \( \tilde{v}_4 \tilde{v}_1 \), \( \tilde{v}_5 \tilde{v}_2 \) and \( \tilde{v}_6 \tilde{v}_3 \) at the same speed. By using
an argument similar to that used in the proof of Theorem 1, we obtain a flat folded octahedron \( Q \) of \( P \) on its face \( \triangle v_1v_2v_3 \) by a continuous folding process of polyhedra \( \{ P_t : 0 \leq t \leq 1 \} \) from \( P \) to \( P_1 \) such that \( P_0 = P \) and \( P_1 = Q \), as shown in Fig. 8(2) where \( f = f_1 \).

Let \( l \) be the line passing through the circumcenters of \( \triangle v_1v_2v_3 \) and \( \triangle v_4v_5v_6 \).

Let \( C = \{ v_1h_1, v_2h_1, v_3h_1 \} \cup \{ h_1p, pv_6, v_2p, v_5p : p \in h_1g_1 \} \) where \( g_1 \) and \( h_1 \) are the circumcenter of \( \triangle v_2v_6v_3 \) and the midpoint of \( v_2v_6 \) respectively, and let \( CT \) be the union of three rotations of \( C \) about \( l \) with radial angles \( 0, 2\pi/3 \) and \( 4\pi/3 \). Then \( P \) is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in \( P \) is \( CT \).

**Theorem 4.** The regular dodecahedron is flattened on its face by a continuous folding process.

Precisely, let \( P \) be the regular dodecahedron with vertices \( v_i \ (1 \leq i \leq 20) \) in \( \mathbb{R}^3 \) as shown in Fig. 9(1), and let \( l \) be the line passing through circumcenters of two regular pentagons with vertices \( v_i \ (1 \leq i \leq 5) \) and \( v_j \ (16 \leq j \leq 20) \). Let \( g_1 \) be the circumcenter of the pentagon \( v_8v_14v_20v_19v_{13} \), and \( h_1 \) be the midpoint of \( v_8v_{13} \). Let \( C = \{ v_1v_8, v_8v_{20}, v_{20}v_{13}, v_{13}v_1, v_1h_1, v_8h_1, v_{13}h_1 \} \cup \{ h_1p, pv_20, v_8p, v_{13}p : p \in h_1g_1 \} \) and \( CT \) be the union of the five rotations of \( C \) about the line \( l \) with radial angles \( 0, 2\pi/5, 4\pi/5, 6\pi/5 \) and \( 8\pi/5 \). Then \( P \) is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in \( P \) is \( CT \).

**Proof.** We denote by \( co(S) \) the convex hull of a set \( S \) in \( \mathbb{R}^3 \). Let \( P \) be the regular dodecahedron with the vertex set \( V = \{ v_i : 1 \leq i \leq 20 \} \) in \( \mathbb{R}^3 \) as shown in Fig. 9(1). We divide the set \( V \) into four subsets \( V_i = \{ v_{5i+j} : 1 \leq j \leq 5 \} \)
Fig. 9. A flat folded regular dodecahedron
Let $g_i (1 \leq i \leq 5)$ be the circumcenters of five pentagonal faces each of which has a common edge with the face $v_{16}v_{17}v_{18}v_{19}v_{20}$, and let $R_i (1 \leq i \leq 5)$ be five folded rhombuses $v_1v_{13}v_{20}v_8$, $v_2v_{14}v_{16}v_9$, $v_3v_{15}v_{17}v_{10}$, $v_4v_{11}v_{18}v_6$ and $v_5v_{12}v_{19}v_7$ included in $P$, as shown in Fig. 9(2).

Denote by $v_{5+i}v_i, (1 \leq i \leq 5)$ the circular arcs with centers $v_5, v_1, v_2, v_3, v_4$, respectively which are drawn on the faces $v_5v_6v_{12}v_7v_1$, $v_1v_7v_{13}v_8v_2$, $v_2v_8v_{14}v_9v_3$, $v_3v_9v_{15}v_{10}v_4$ and $v_4v_{10}v_{11}v_6v_5$, respectively. Apply Proposition 3 to those folded rhombuses simultaneously such that $v_i (6 \leq i \leq 10)$ moves from $v_{5+i}$ to $v_i$ along $v_{5+i}v_i$ at the same speed for all $6 \leq i \leq 10$.

We construct a continuous folding process for polyhedra $\{P_t: 0 \leq t \leq 1\}$ with piecewise linear mappings $f_t$ as follows.

Step 1. For $v_i (1 \leq i \leq 5)$, define $f_t(v_i) = v_i$ for all $0 \leq t \leq 1$.

Step 2. Five vertices $v_i (6 \leq i \leq 10)$ move along the circular arcs $v_{5+i}v_i$ with centers $v_i$ ($i = 5, 1, 2, 3, 4$), respectively at the same speed as $t$ increases from 0 to 1. Let $f_t(v_i) (6 \leq i \leq 10)$ be the trace of $v_i$ for each $0 \leq t \leq 1$, as suggested in Fig. 9(3). Then $f_t(v_i) (6 \leq i \leq 10)$ are at the vertices of a smaller rotated regular pentagon similar to the pentagon with vertices $v_i (6 \leq i \leq 10)$.

Step 3. For each $0 \leq t \leq 1$, let $S_t$ be the cylinder which includes the circumscribed circle of the regular pentagon with vertices $f_t(v_i) (6 \leq i \leq 10)$. Then there is a unique point $p \in S_t$ such that

$$|v_1p| = |v_1v_{13}|, |f_t(v_7)p| = |v_7v_{13}|, |f_t(v_8)p| \leq |v_8v_{13}|.$$

Define $f_t(v_{13}) = p \in S_t$.

Step 4. Let $i = 11, 12, 14$ or 15. For each $0 \leq t \leq 1$, define $f_t(v_i)$ similarly to $f_t(v_{13})$ by the point on the cylinder $S_t$ such that $\{f_t(v_i): 11 \leq i \leq 15\}$ are at the vertices of a congruent rotated pentagon of the pentagon with vertices $\{f_t(v_i): 6 \leq i \leq 10\}$.

Step 5. Let $16 \leq i \leq 20$. Let $T_i$ be the cylinder which includes the circumscribed circle of the regular pentagon $v_1v_2v_3v_4v_5$. Define $f_t(v_i) (16 \leq i \leq 20)$ satisfying the following:

1. $f_t(v_i) (16 \leq i \leq 20)$ are at the vertices of a congruent parallel pentagon to the pentagon $v_1v_2v_3v_4v_5$, which is included in $T_i$.
2. The convex hull of 10 vertices $\{f_t(v_i): 11 \leq i \leq 20\}$ is congruent to that of ten vertices $\{f_t(v_i): 1 \leq i \leq 10\}$.
3. $|c_0c_{i}| = |c_0c_1| + |c_1c_{i}|$, where for each $0 \leq i \leq 3$ $c_i$ is the center of the regular pentagon with vertices $\{f_t(v_{5+k}): 1 \leq k \leq 5\}$.

Step 6. For each $0 \leq t \leq 1$, extend $f_t$ to all points in the folded rhombuses $R_i (1 \leq i \leq 5)$ which is defined by Proposition 3, and then extend it again to all points in $P$ as a piecewise isometric mapping. Define $P_t = \{f_t(p): p \in P\}$.
for $0 \leq t \leq 1$. Then $P_0 = P$, and $P_1$ is a flat folded polyhedron of $P$ which is a multi-covered face of $P$.

Now, we obtained a continuous folding process for polyhedra $\{P_t : 0 \leq t \leq 1\}$ with piecewise isometric mappings $\{f_t : 0 \leq t \leq 1\}$ from $P$ to $P_t$ such that $P_0 = P$, and that $P_1$ is a multi-covered face on the regular pentagon $\{v_1v_2v_3v_4v_5\}$ of $P$, as shown in Fig. 9(4).

Therefore, $P$ is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in $P$ is $C_T$, where $C_T$ is the set defined in the statement of Theorem 4. This completes the proof of Theorem 4.

\[\square\]

**Theorem 5.** The regular icosahedron is flattened on its face by a continuous folding process.

Precisely, Let $P$ be the regular icosahedron with vertices $v_i (1 \leq i \leq 12)$ in $\mathbb{R}^3$, $g_i (1 \leq i \leq 3)$ be circumcenters of $\triangle v_2v_3v_5$, $\triangle v_5v_9v_{12}$ and $\triangle v_8v_5v_{12}$ respectively, as shown in Figure 10(1)(2). Let $h_i (1 \leq i \leq 3)$ be midpoints of $v_5v_9$, $v_8v_5$ and $v_8v_9$ respectively. Let

\[
C_1 = \{v_1h_1, v_2h_1, v_5h_1\} \cup \{h_1p, pv_9, v_5p : p \in h_1g_1\},
\]

\[
C_2 = \{v_1h_2, v_9h_2, v_12h_2\} \cup \{h_2p, pv_5, v_12p : p \in h_2g_2\},
\]

\[
C_3 = \{v_1h_3, v_5h_3, v_8h_3\} \cup \{h_3p, pv_{12}, v_5p : p \in h_3g_3\},
\]

and $C = C_1 \cup C_2 \cup C_3$. Let $l$ be the line passing through the circumcenters of $\triangle v_1v_2v_3$ and $\triangle v_1v_9v_{11}v_{12}$. Define $C_T$ to be the join of three rotations of $C$ about $l$ with radial angles $0$, $2\pi/3$ and $4\pi/3$. Then $P$ is flattened on its face by a continuous folding process for polyhedra, for which the crease pattern in $P$ is $C_T$.

**Proof.** Let $P$ be the regular icosahedron with vertices $v_i (1 \leq i \leq 12)$ in $\mathbb{R}^3$, as in Fig. 10(1). We divide the set of twelve vertices of $P$ into four subsets $V_i = \{v_{3i+j} : 1 \leq j \leq 3\}$ ($0 \leq i \leq 3$), and denote by $\triangle V_i$ the triangle with vertices in $V_i$. Then $\triangle V_0$ and $\triangle V_3$ (also $\triangle V_1$ and $\triangle V_2$) are congruent equilateral triangles. In order to obtain a flat folded polyhedron of $P$ on its face, we show that there is a continuous folding process for polyhedra $\{P_t : 0 \leq t \leq 1\}$ such that images of $\triangle V_0$ and $\triangle V_3$ (also $\triangle V_1$ and $\triangle V_2$) by $f_t$ are congruent equilateral triangles in $P_t$.

Let $g_i (1 \leq i \leq 9)$ be the circumcenters of nine triangular faces of $P$ ($\triangle v_2v_3v_9$, $\triangle v_3v_6v_7$, $\triangle v_6v_4v_8$, $\triangle v_5v_9v_{12}$, $\triangle v_6v_7v_{10}$, $\triangle v_4v_8v_{11}$, $\triangle v_8v_5v_{12}$, $\triangle v_9v_6v_{10}$, and $\triangle v_7v_4v_{11}$), as shown in Fig. 10(2) for $g_1$, $g_2$ and $g_3$. Let $E$ be the set of nine edges

\[
E = \{v_iv_{i+3}, v_{i+3}v_{i+6}, v_{i+6}v_{i+9} : 1 \leq i \leq 3\}.
\]

Let $R_i (1 \leq i \leq 9)$ be nine folded rhombuses each of which consists of two faces of $P$ containing a common edge in $E$, that is, $v_1v_2v_9v_5$, $v_2v_3v_7v_6$, $v_3v_1v_8v_4$, $v_1v_{10}v_{11}v_4v_7$, $v_{12}v_{11}v_5v_8$, $v_1v_5v_{12}v_8$, $v_2v_6v_{10}v_9$ and $v_3v_4v_{11}v_7$.

By using an argument similar to that used in the proof of Theorem 4, we can define a continuous folding process for polyhedra $\{P_t : 0 \leq t \leq 1\}$ with piecewise linear mappings $\{f_t : 0 \leq t \leq 1\}$ from $P$ into $\mathbb{R}^3$ such that $P_0 = P$, and $P_1$ is a multi-covered face of $\triangle v_1v_2v_3$ of $P$, as suggested in Fig. 10(3)(4). We omit the details of the proof. \[\square\]
Fig. 10. A flat folded regular icosahedron
4  Further Research

We showed each Platonic polyhedron $P$ is flattened on its face by a continuous folding process whose crease pattern is 2-dimensional set. What is the minimum area of creases for such process? Denote $r = (\text{the area of creases}) / (\text{the area of } P)$. Our examples shown in Theorems 1 through 5 have $r = 1/12$ for a regular tetrahedron, $r = 1/6$ for a cube, $r = 1/8$ for a regular octahedron, $r = 1/12$ for a regular dodecahedron, and $r = 3/20$ for a regular icosahedron.

In [4] p. 281, it says that we can find a flat folded state of a polyhedral surface that is homeomorphic to a disk or a sphere. In this paper we worked on Platonic polyhedra. By using our results, we have shown that if a polyhedron $Q$ is packed by finite congruent copies of a Platonic polyhedron (except the tetrahedron) in one direction in a face-to-face manner, then $Q$ is flattened on its face by a continuous folding process for polyhedra.

References